On the propagation speed of photons in axion dark matter

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Refs: AI, Teruaki Suyama, arXiv: 2210.15213 (2022)

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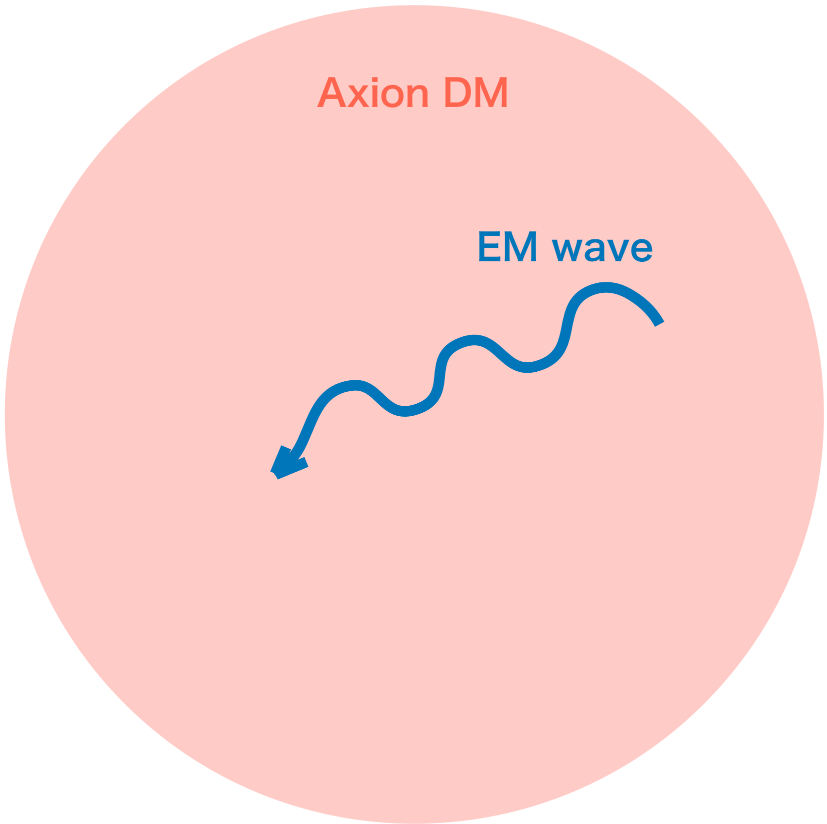
Motivation

- Axion (or axion like particle) is the strong candidate for the dark matter
- Photons can interact with the axion through the Chern-Simon term $\frac{1}{4}a(x)g_{a\gamma\gamma}F_{\mu\nu}\tilde{F}^{\mu\nu}$

Consider photons propagating in the axion dark matter background



EoM is $\Box \boldsymbol{A}(\boldsymbol{x}, t) + g_{a\gamma\gamma}\dot{a}\nabla \times \boldsymbol{A} = 0$. We will assume $g_{a\gamma\gamma}\dot{a} = \theta = \mathrm{const.}$ for simplicity.

In the Fourier space, the dispersion relation is $\boxed{\omega^2 = k^2 \pm \theta k}$

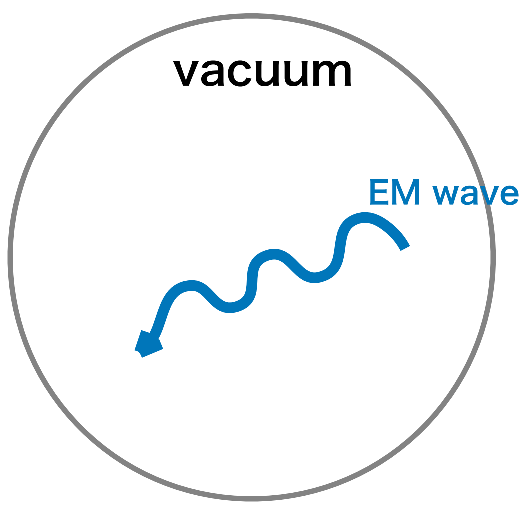
(+,- are for the two circular polarization modes)

Motivation

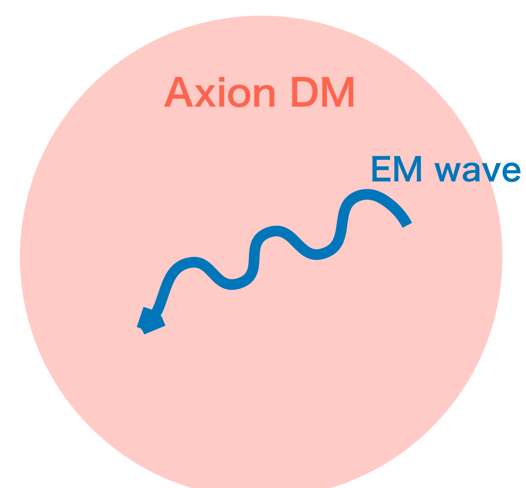
From the dispersion relation, $\boxed{\omega^2 = k^2 \pm \theta k}$

The group velocity is

$$\boxed{\frac{d\omega}{dk} = \frac{k \pm \theta/2}{\sqrt{k^2 \pm k\theta}} \simeq 1 + \frac{1}{8}\frac{\theta^2}{k^2} \mp \frac{1}{8}\frac{\theta^3}{k^3} + \cdots > 1}$$



$$\frac{d\omega}{dk} = c$$



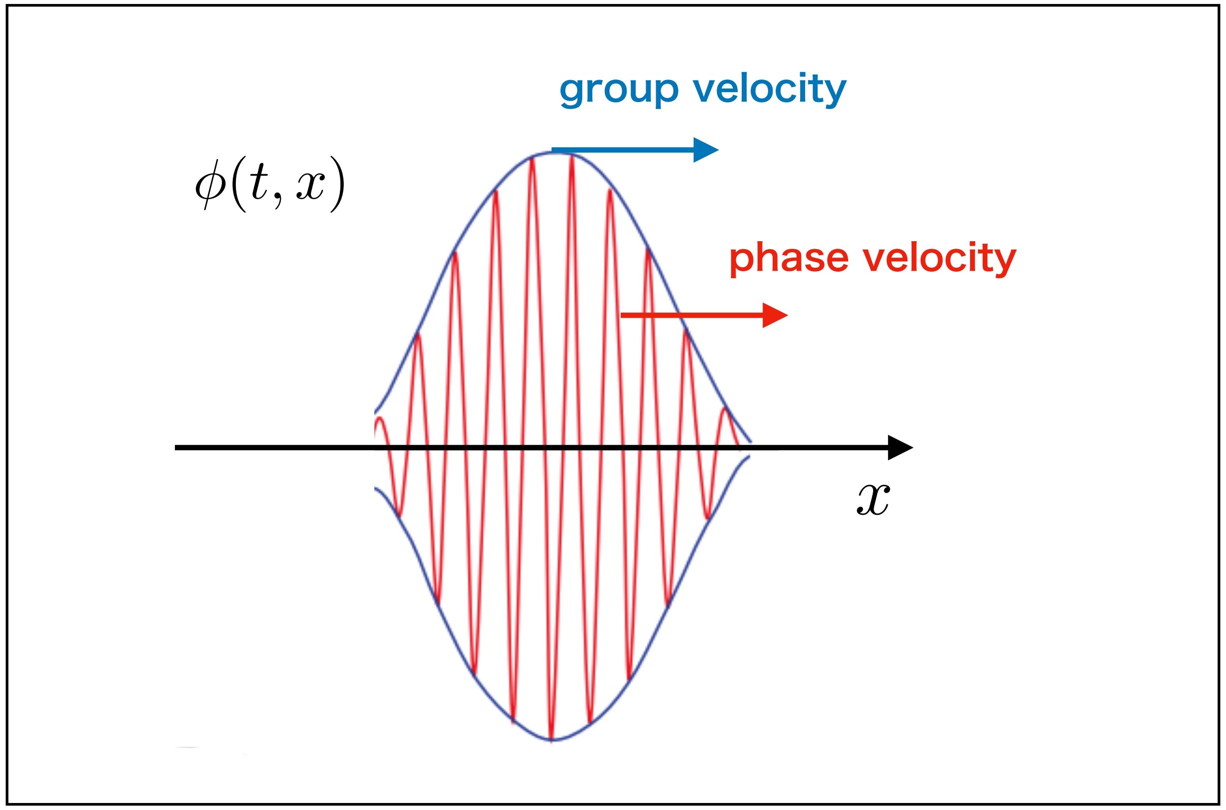
$$\frac{d\omega}{dk} > c$$

What does this mean?

Can information propagate superluminally in the axion DM background…?

➡ No! We need to reconsider what group velocities are.

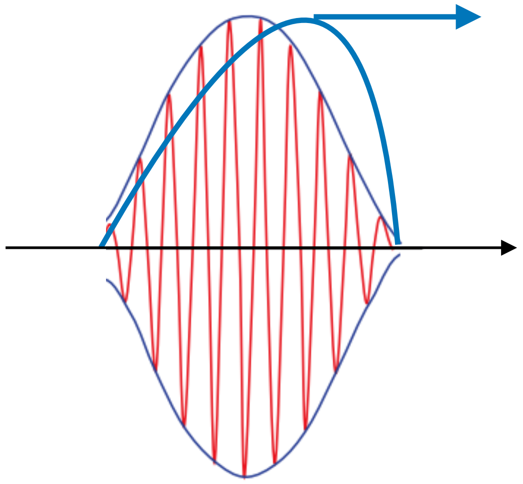
Propagation speed



If the phase velocity depends on k , the envelope of the wave packet is distorted.

axion case

$$\frac{\omega}{k} = \sqrt{1 \pm \frac{\theta}{k}}$$

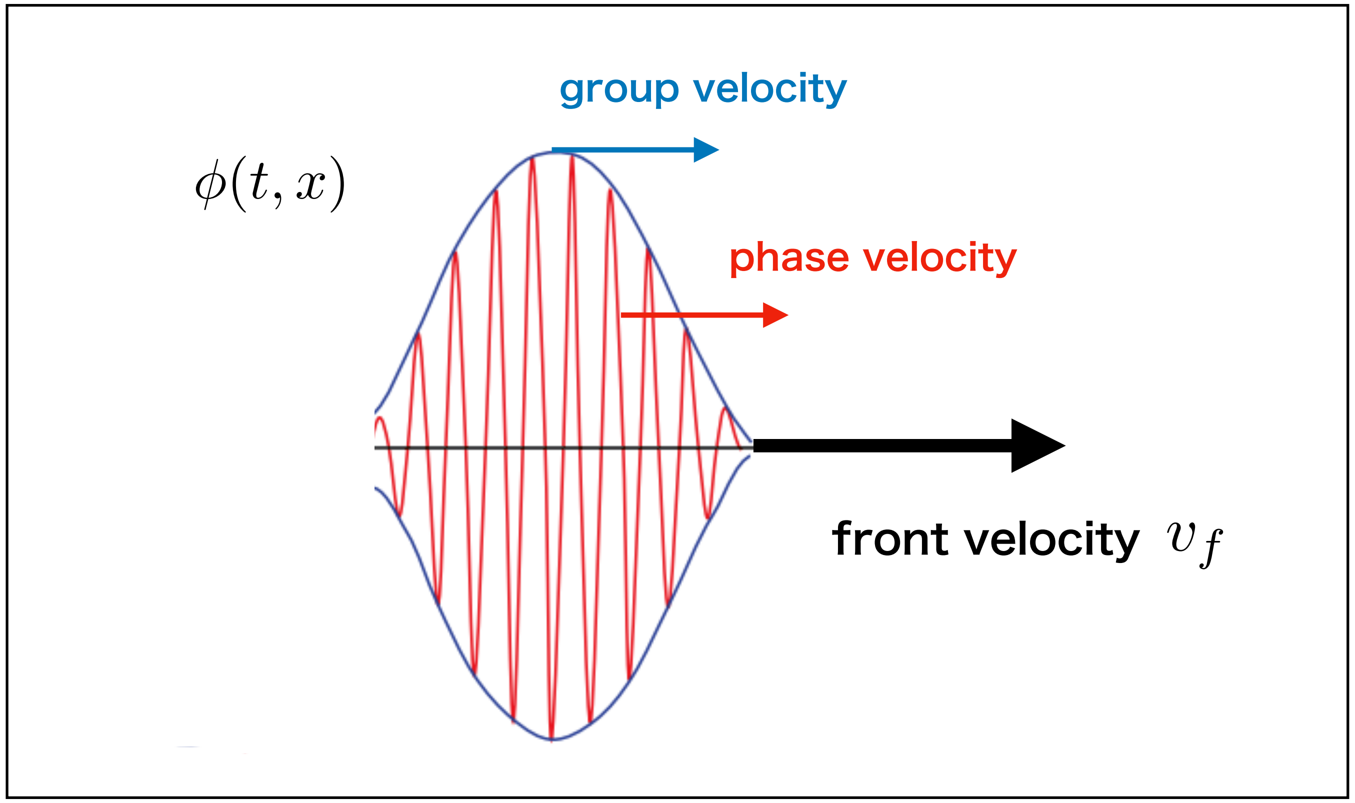


The group velocity can be superluminal.
Actually it has been observed for photons in materials

(Withawat. W, et.al, 10.1109/JPROC.2010.2052910)

The group velocity is not an appropriate measure to investigate the propagation speed.
What is the appropriate one?

Propagation speed



The front velocity is the appropriate measure to investigate if the wave actually propagate superluminally or not.

If $v_f > c$, it is superluminal

➡ Let us study the front velocity of photons in the axion dark matter

Talk plan

1. We will prove the relation $\boxed{v_f = \lim_{k\to\infty} v_p(k)}$ for the dispersion relation of $\omega(k)^2 = k^2 \pm k^\alpha M$

(L. I. Mandelstam, (1972))

$(M : \text{const.},\ \alpha = 0, 1, 2)$

Therefore, for photons propagating in the axion DM, $\omega^2 = k^2 \pm \theta k$, the front velocity is

$$v_f = \lim_{k\to\infty} c \pm \frac{\theta}{k} = c$$ $\Rightarrow$ not superluminal

$\boxed{v_f = \lim_{k\to\infty} v_p(k)}$ implies that only UV physics determines the propagation speed and IR physics is irrelevant…?

2. We will investigate how much powerful the relation is by considering more general dispersion relations.

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Front velocity and phase velocity

Let us prove $\boxed{v_f = \lim_{k\to\infty} v_p(k)}$ for the second order partial differential equation (PDE).

Any second order PDE for $u(t,x)$ can be rewritten by a system of first order PDEs as

$$\boxed{a_{ij}\frac{\partial \phi_j}{\partial t} + b_{ij}\frac{\partial \phi_j}{\partial x} + c_{ij}\phi_j = 0} \quad \cdots\cdots\cdots\cdots \quad ①$$

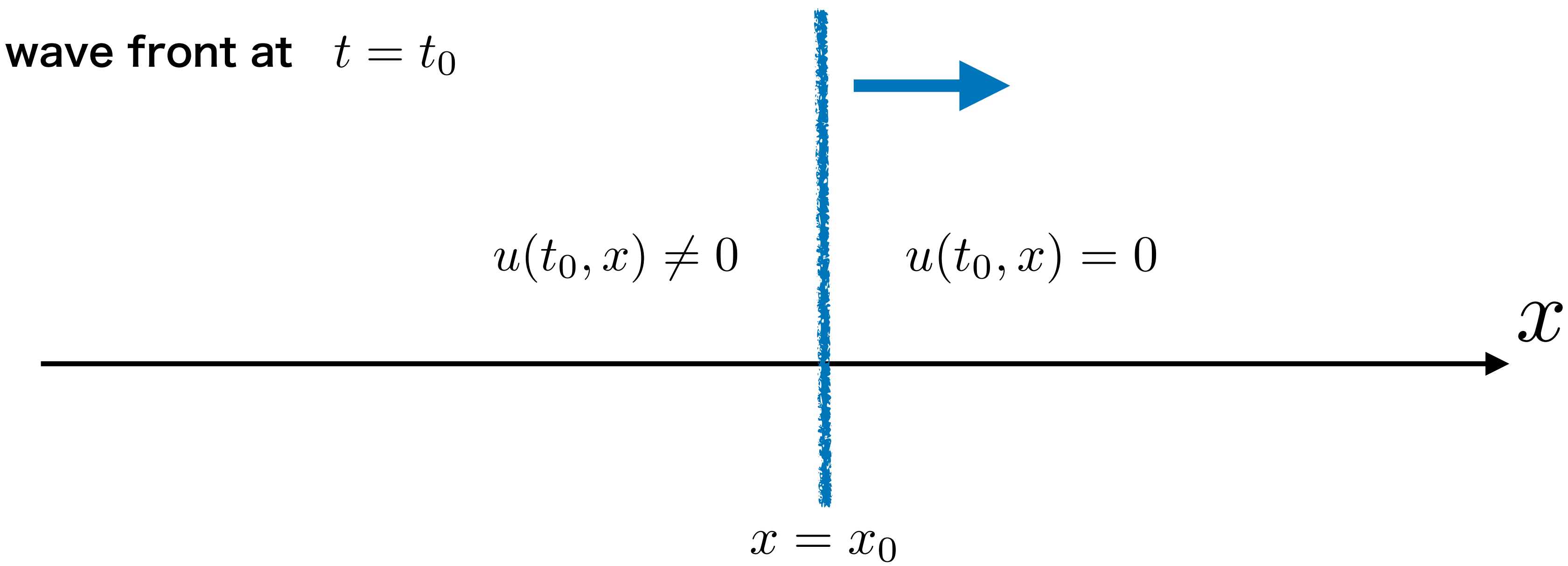
where $\phi_i = \left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\right)$

ex.)

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + m^2 u = 0 \quad \Longrightarrow \quad \left(\begin{array}{l} \dfrac{\partial \phi_1}{\partial t} = \phi_2 \\[2ex] \dfrac{\partial \phi_2}{\partial t} = \dfrac{\partial \phi_3}{\partial x} - m^2\phi_1 \\[2ex] \dfrac{\partial \phi_3}{\partial t} = \dfrac{\partial \phi_2}{\partial x} \end{array} \right.$$

Front velocity and phase velocity

We now consider the wave front at $t = t_0$



Consider the curve tracing the wave front. At (t_0, x_0) on the characteristic, we have

$$\left.\frac{d\phi_i}{dt}\right|_0 = \left.\frac{\partial \phi_i}{\partial t}\right|_0 + \left.\frac{\partial \phi_i}{\partial x}\right|_0 \left.\frac{dx}{dt}\right|_0 \quad \cdots\cdots\cdots\cdots \text{②}$$

front velocity v_f

$$\left(\boxed{a_{ij}\frac{\partial \phi_j}{\partial t} + b_{ij}\frac{\partial \phi_j}{\partial x} + c_{ij}\phi_j = 0} \quad \cdots\cdots\cdots\cdots \text{①} \right)$$

Substituting ② into ① yields $\displaystyle (-a_{ij}v_f + b_{ij}) \left.\frac{\partial \phi_j}{\partial x}\right|_0 + a_{ij}\left.\frac{d\phi_j}{dt}\right|_0 + c_{ij}\phi_j^{(0)} = 0$

Since ϕ_i is not determined or not unique on the characteristic line, we require the condition:

$$\boxed{\det(a_{ij}v_f = b_{ij}) = 0 \quad \cdots\cdots\cdots\cdots \text{③}}$$

Front velocity and phase velocity

On the other hand, using an ansatz, $\phi_i = \varphi_i e^{i(kx-\omega(k)t)}$ in ①, we obtain

$$\left(\boxed{a_{ij}\frac{\partial \phi_j}{\partial t} + b_{ij}\frac{\partial \phi_j}{\partial x} + c_{ij}\phi_j = 0} \quad \cdots\cdots\cdots\cdots ① \right)$$

$$\big(i\omega(k)a_{ij} - ikb_{ij} + c_{ij}\big)\varphi_j = 0$$

In order to have a non-trivial solution, we have

$$\det(a_{ij}\frac{\omega(k)}{k} - b_{ij} - \frac{i}{k}c_{ij}) = 0 \quad \cdots\cdots\cdots\cdots ④$$

phase velocity $v_p(k)$

$$\left(\boxed{\det(a_{ij}v_f = b_{ij}) = 0 \quad \cdots\cdots\cdots\cdots ③} \right)$$

Comparing ③ and ④, we find

$$\boxed{v_f = \lim_{k\to\infty} v_p(k)}$$

Front velocity and phase velocity

The relation, $\boxed{v_f = \lim_{k\to\infty} v_p(k)}$ implies that only UV physics is important for the front velocity.

ex.) photons in axion dark matter

dispersion relation: $\omega^2 = k^2 \pm \theta k$ $\longrightarrow$ $v_f = \lim_{k\to\infty} \sqrt{c \pm \frac{\theta}{k}} = c$

not superluminal

- How common the relation is?
  UV physics always determines the propagation speed and IR physics is irrelevant at all?

※ The proof can not apply, for example, to the wave equation with higher derivatives.

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^3 u}{\partial x^3} = 0 \quad \Longrightarrow \quad \phi_i = \left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right)$$

$$\downarrow$$

$$\left(\frac{\partial \phi_1}{\partial t} = \phi_2 , \quad \frac{\partial \phi_2}{\partial t} = \frac{\partial \phi_3}{\partial x} + \frac{\partial \phi_4}{\partial x} , \quad \frac{\partial \phi_3}{\partial t} = \frac{\partial \phi_2}{\partial x} , \quad \cancel{\frac{\partial \phi_4}{\partial t} = \frac{\partial^2 \phi_2}{\partial x^2}} . \right)$$

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2. We will investigate how much powerful the relation is by considering more general dispersion relations.

Front velocity

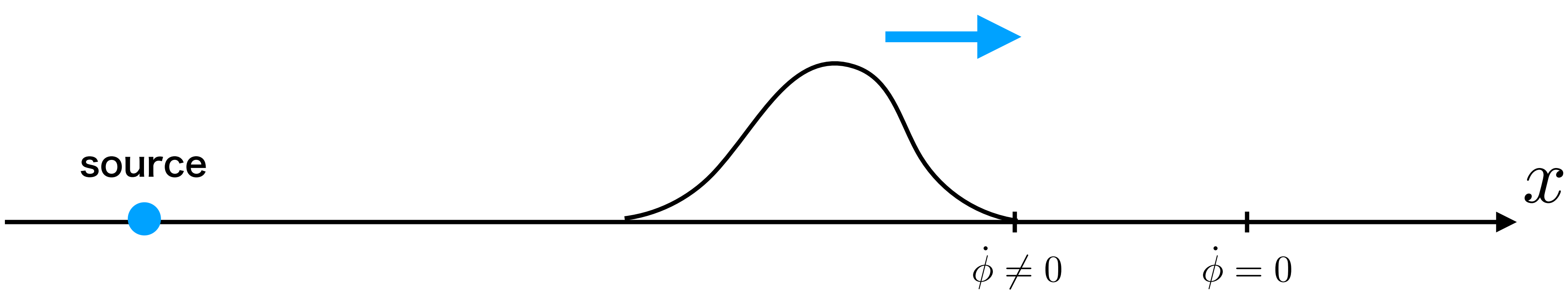
To investigate the front velocity to a given dispersion relation, we consider the Green function.
The wave equation with a point source is

$$\left(-\frac{\partial^2}{\partial t^2} + \omega^2(-\partial_x^2)\right)\phi(t,x) = -\delta(t)\delta(x).$$

Then, the retarded Green function is given by

$$\phi(t,x) = \theta(t)\int \frac{dk}{2\pi} e^{ikx}\frac{\sin(\omega_k t)}{\omega_k}.$$

The wave front is defined as the point where $\dot{\phi}$ has just become non-zero for the first time.

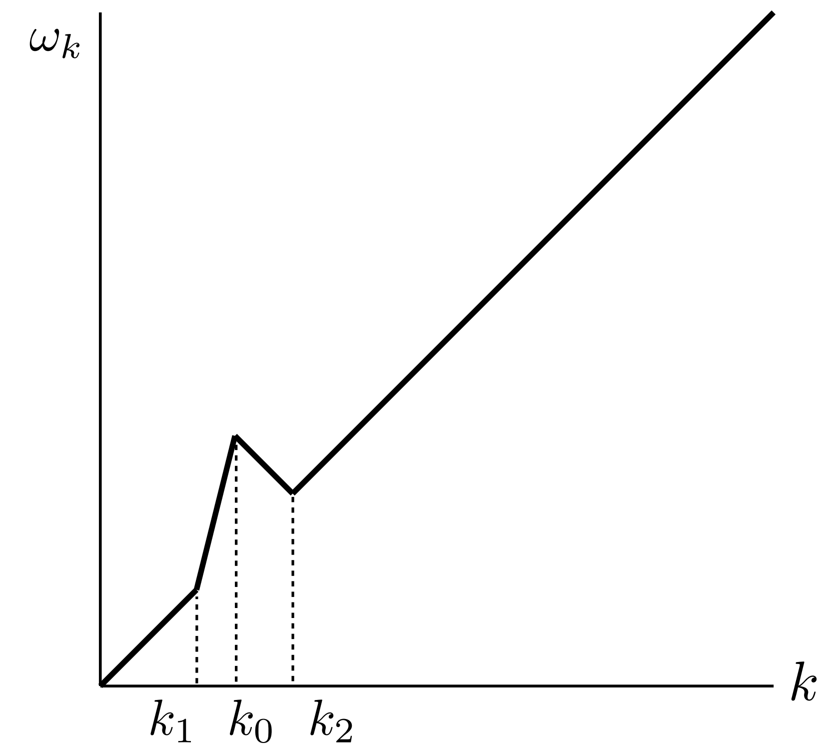


For $t > 0$, $\boxed{\dot{\phi}(t,x) = \int \frac{dk}{2\pi} e^{ikx}\cos(\omega_k t)}$

We will evaluate the $\dot{\phi}$ for a given dispersion relation.

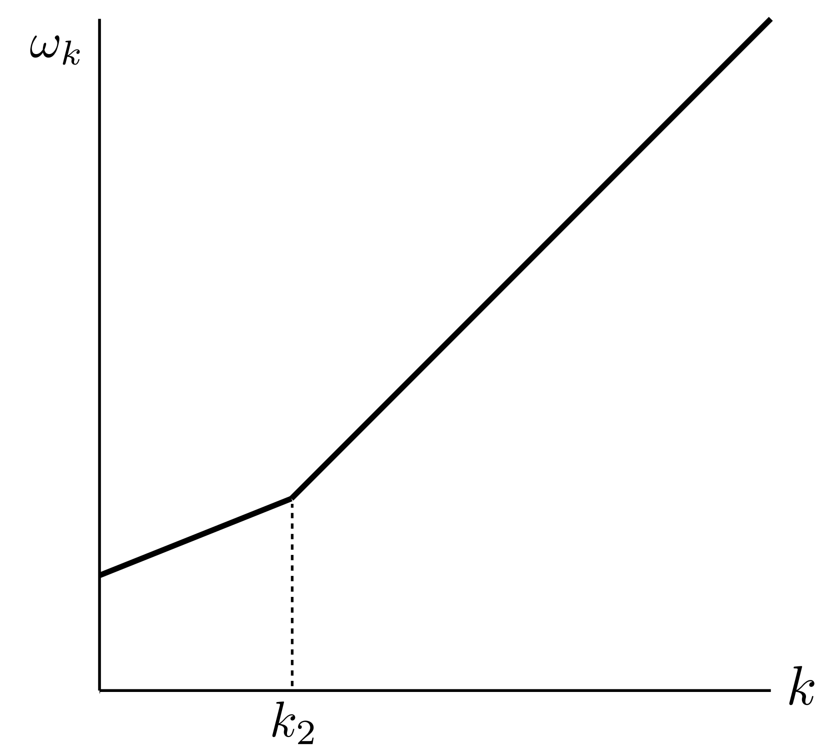
Cuspy modulation

We first consider a dispersion which has a cusp in IR regime



$$\omega_k = \begin{cases} k & \text{for} \quad 0 < k < k_1, \quad k \geq k_2 \quad , \\ ak + (1-a)k_1 & \text{for} \quad k_1 \leq k < k_0 \quad , \\ bk + (1-b)k_2 & \text{for} \quad k_0 \leq k < k_2 \quad , \end{cases}$$

where $b = \frac{k_2 - k_1 - a(k_0 - k_1)}{k_2 - k_0}$



Especially, when $k_1 = k_0 = 0$,

it resembles the standard massive dispersion relation

$$\left(\omega_k = \sqrt{k^2 + m^2}\right)$$

Cuspy modulation

For the cuspy dispersion relation,

$$\omega_k = \begin{cases} k & \text{for} \quad 0 < k < k_1, \quad k \geq k_2 \quad , \\ ak + (1-a)k_1 & \text{for} \quad k_1 \leq k < k_0 \quad , \\ bk + (1-b)k_2 & \text{for} \quad k_0 \leq k < k_2 \quad , \end{cases}$$

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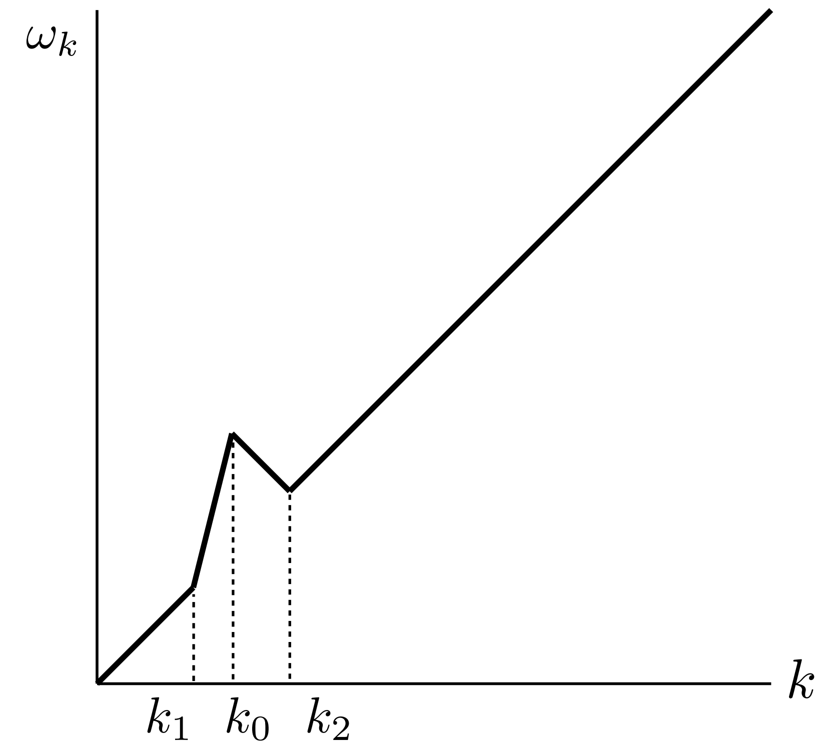


One can carry out $\boxed{\dot{\phi}(t,x) = \int \frac{dk}{2\pi} e^{ikx} \cos(\omega_k t)}$ analytically as

$$\boxed{\begin{aligned} \dot{\phi}(t,x) = & \frac{1}{2}\left[\delta(x-t) + \delta(x+t)\right] \longleftarrow \text{luminal propagation for } \omega = k \\ & + \frac{(t+x)\left\{ \sin\left[k_1(t-x)\right] - \sin\left[k_2(t-x)\right] \right\} + (x \to -x)}{2\pi(t^2 - x^2)} \\ & + \frac{(at+x)\left\{ \sin\left[k_0(at-x) + k_1(1-a)t\right] - \sin\left[k_1(t-x)\right] \right\} + (x \to -x)}{2\pi(a^2t^2 - x^2)} \\ & - \frac{(bt+x)\left\{ \sin\left[k_0(bt-x) + k_2(1-b)t\right] - \sin\left[k_2(t-x)\right] \right\} + (x \to -x)}{2\pi(b^2t^2 - x^2)} \ . \end{aligned}}$$

We see that $\dot{\phi}$ is non-zero even in spacelike region $x > t$. ➡ superluminal!

General small modulation

We now consider general modulation to the standard dispersion relation of a massive particle

$$\omega_k^2 = k^2 + \delta\tilde{\omega}(k)^2$$

Assuming that the modulation is small for arbitrary k, i.e., $\delta\tilde{\omega}(k) \ll k$, one can obtain

$$\omega_k = \sqrt{k^2 + \delta\tilde{\omega}(k)^2} \simeq k^2 + \delta\omega \ ,$$

where $\delta\omega(k) \equiv \delta\tilde{\omega}^2(k)/2k \ll k$ and the parity conservation is imposed; $\delta\omega(-k) = \delta\omega(k)$.

Then, $\dot{\phi}$ is

$$\boxed{\begin{aligned}\dot{\phi}(t,x) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \cos\left((k + \delta\omega(k))t\right) \\ &\simeq \underline{\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \cos(kt)} - \int_0^{\infty} \frac{dk}{\pi} \cos(kx)\sin(kt)\delta\omega(k)t.\end{aligned}}$$

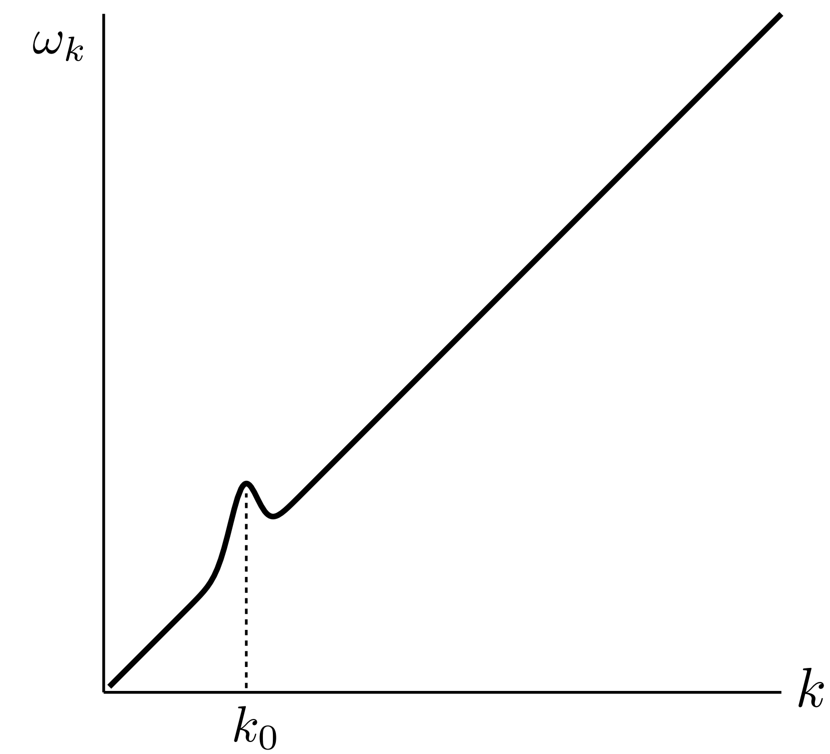
luminal propagation

$$= \frac{1}{2}\left[\delta(x-t) + \delta(x+t)\right]$$

Bumpy modulation

For example, if we consider a Gaussian-like function for the modulation,

$$\boxed{\delta\omega(k) = A\exp\left(-\frac{(k-k_0)^2}{2\sigma^2}\right)}$$



One can analytically evaluate $\dot{\phi}$ as

$$\begin{aligned}\dot{\phi}(t,x) &= \int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ikx}\cos\left((k+\delta\omega(k))t\right)\\ &\simeq \int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{ikx}\cos(kt) - \int_0^{\infty}\frac{dk}{\pi}\cos(kx)\sin(kt)\delta\omega(k)t.\\ &= \frac{1}{2}\left[\delta(x-t)+\delta(x+t)\right]\\ &\quad -\frac{\sigma At}{\sqrt{2\pi}}\left(e^{-\sigma^2(t+x)^2/2}\sin\left[(t+x)k_0\right] + e^{-\sigma^2(t-x)^2/2}\sin\left[(t-x)k_0\right]\right).\end{aligned}$$

non-zero even in spacelike region $x > t$

superluminal!

General small modulation

As we saw, for the general small modulation to the dispersion relation,

$$\boxed{\omega_k = \sqrt{k^2 + \delta\tilde{\omega}(k)^2} \simeq k^2 + \delta\omega}$$

The wave front is

$$\begin{aligned}\dot{\phi}(t,x) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \cos\left((k+\delta\omega(k))t\right) \\ &\simeq \underbrace{\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \cos(kt)} - \underbrace{\int_{0}^{\infty} \frac{dk}{\pi} \cos(kx)\sin(kt)\delta\omega(k)t}.\end{aligned}$$

luminal propagation

$$= \frac{1}{2}\left[\delta(x-t) + \delta(x+t)\right]$$

In general, without special cancellation, this is non-zero for $x > t$

➡ For the general modulation in IR or intermediate energy scale, the propagation speed would be superluminal.

Summary

- Photons in the axion dark matter apparently propagate superluminally (group velocity > c)

- We proved $\boxed{v_f = \lim_{k\to\infty} v_p(k)}$ and showed that the front velocity is luminal for the axion case.

  (L. I. Mandelstam, (1972))

→ The Fourier space is not appropriate to discuss the propagation speed.
(This may have important Implication on the causality discussion, such as in EFT)

- We examined how much powerful $\boxed{v_f = \lim_{k\to\infty} v_p(k)}$ is in general dispersion relations.

→ In general, IR property of dispersion relations affects the propagation speed and superluminal propagation is realized easily.
(The Lorentz invariance is significant...?)